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Gravitational perturbation and quasi-normal modes of charged black holes in Einstein-Born-Infeld gravity

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Abstract

Born-Infeld electrodynamics has attracted considerable interest due to its relation to strings and D-branes. In this paper the gravitational perturbations of electrically charged black holes in Einstein-Born-Infeld gravity are studied. The effective potentials for axial perturbations are derived and discussed. The quasi normal modes for the gravitational perturbations are computed using a WKB method. The modes are compared with those of the Reissner-Nordstrom black hole. The relation of the quasi normal modes with the non-linear parameter and the spherical index are also investigated. Comments on stability of the black hole and on future directions are made.

Key words: Static, Charged, Black Holes, Born-Infeld, Quasinormal modes

1 Introduction

Born-Infeld electrodynamics was first introduced in 1930's to obtain a finite energy density model for the electron [1]. It has attracted considerable interest in recent times due to various reasons. One of the motivations being the observations that it arises naturally in open superstrings and in D-branes [2]. The low energy effective action for an open superstring in loop calculations lead to Born-Infeld type actions [3]. It has also been observed that the Born-Infeld action arises as an effective action governing the dynamics of vector-fields on D-branes [4]. For a review of aspects of Born-Infeld theory in string theory see Gibbons [5].

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In this paper the main focus is to study the gravitational perturbations and the quasi-normal modes of static charged black hole in Einstein-Born-Infeld gravity. The properties of this black hole is discussed in [6] [7].

Perturbations of black holes lead to damped oscillations. The modes of such oscillations are called quasi-normal modes(QNM). The frequencies of QNM's depend on the parameters of the black holes such as mass, charge and angular momentum and are independent of initial perturbations. If the radiation due to QNM modes are detected in the future by gravitational wave detectors, it would be a clear way of identifying the possible charges of black holes. There are extensive studies of QNM's in various black-hole backgrounds in the literature. See the review by Kokkotas et. al. [8] for more information.

There are many reasons to study QNM's. One of them is Loop Quantum Gravity. It has been observed that for asymptotically flat black holes, the real part of the high overtones of QNM's coincide with the Barberr-Immirizi parameter γ. The value of γ fixed via QNM's turned out to be precisely the one required to make the Loop Quantum Gravity entropy predictions coincide with the classical Bekenstien-Hawking entropy. Some of the work done along these lines are given in the references [9] [10] [11] [12] [13] [14] [15] [16] [17] [18].

Another important aspect of QNM studies have been related to the conjecture relating anti-de Sitter(AdS) and conformal field theory (CFT) [19]. It is conjectured that the imaginary part of the QNM's which gives the time scale to decay the black hole perturbations corresponds to the time scale of the CFT on the boundary to reach thermal equilibrium. There are many work on AdS black holes in four and higher dimensions on this subject such as [20][21] [22] [23][24] [25] [26] [27] [28] [29] [30] [31] [32] [33] [34] [35].

The paper is presented as follows: In section 2 the black hole solutions are introduced. In section 3 the gravitational perturbations are given. In section 4 the QNM's are computed and discussed. Finally, the conclusion is given in section 5.

2 Static charged black hole in Einstein-Born-Infeld gravity

In this section an introduction to the static charged black hole in Einstein-Born-Infeld gravity is given. The most general action for a theory with non-linear electrodynamics coupled to gravity is as follows:

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{16\pi G} + L(F)\right] \tag{1}$$

Here, $L(F)$ is a function of the field strength $F_{\mu\nu}$ only. In the weak field limit, $L(F)$ has to be of the form

$$L(F) = -F^{\mu\nu}F_{\mu\nu} + O(F^4) \tag{2}$$

In this paper, a particular non-linear electrodynamics called Born-Infeld theory is studied which has attracted lot of attention due to its relation to string effective actions. The function $L(F)$ for Born-Infeld electrodynamics may be expanded to be

$$L(F) = 4\beta^2 \left(1 - \sqrt{1 + \frac{F^{\mu\nu}F_{\mu\nu}}{2\beta^2}}\right) \tag{3}$$

Here, β has dimensions $length^{-2}$ and G $length^2$. In the following sections it is assumed that $16\pi G = 1$. Note that when $\beta \to \infty$, the Lagrangian $L(F)$ approaches the one for Maxwell's electrodynamics given by $-F^2$.

To describe the equations of motion for the electrodynamic field strength, a second rank tensor $G^{\mu\nu}$ is defined as,

$$G^{\mu\nu} = \frac{F^{\mu\nu}}{\sqrt{1 + \frac{F^2}{2\beta^2}}} \tag{4}$$

By extremising $L(F)$, with respect to A_μ, one obtain the equations of motion,

$$\bigtriangledown_\mu G^{\mu\nu} = 0 \Rightarrow d * G = 0 \tag{5}$$

The field strength also satisfy the Bianchi identity,

$$\bigtriangledown_{[\mu} F_{\nu\theta]} = 0 \Rightarrow dF = 0 \tag{6}$$

Note that the Bianchi identity is satisfied since the field strength can be written as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. On the other hand, $G_{\mu\nu}$ does not satisfy the Bianchi identity. In comparison, the equations for Maxwell electrodynamics are,

$$\bigtriangledown_\mu F^{\mu\nu} = 0 \Rightarrow d * F = 0 \quad \text{and} \quad \bigtriangledown_{[\mu} F_{\nu\theta]} = 0 \Rightarrow dF = 0 \tag{7}$$

For static spherically symmetric case,

$$G_{tr} = -\frac{Q}{r^2} \quad \text{or} \quad F_{tr} = E(r) = -\frac{Q}{\sqrt{r^4 + \frac{Q^2}{\beta^2}}} \tag{8}$$

Note that for this case the non-linear Lagrangian reduces to,

$$L(F) = 4\beta^2 \left(1 - \sqrt{1 - \frac{E^2}{\beta^2}}\right) \tag{9}$$

imposing an upper bound for $|E| \leq \beta$. This is a crucial characteristic of Born-Infeld electrodynamics which leads to finite self energy of the electron.

By varying the total Lagrangian in eq.(1) with respect to $g_{\mu\nu}$ and assuming $16\pi G = 1$, the gravitational field equations are obtained as,

$$R_{\mu\nu} = T_{\mu\nu} \tag{10}$$

where,

$$T_{\mu\nu} = -2\left(\frac{g^{\alpha\theta}F_{\mu\alpha}F_{\nu\theta}}{\sqrt{1+\frac{F^2}{2\beta^2}}} + \frac{g_{\mu\nu}L(F)}{4}\right) \tag{11}$$

Note that for $\beta \to \infty$, $T_{\mu\nu}$ approaches the energy-momentum tensor for the Maxwell's electrodynamics given by,

$$T_{\mu\nu} = -2\left(g^{\alpha\theta}F_{\mu\alpha}F_{\nu\theta} - \frac{g_{\mu\nu}}{4}F^2\right) \tag{12}$$

By solving the field equations, the static charged black hole with spherical symmetry can be obtained as,

$$ds^2 = f(r)dt^2 - f(r)^{-1}dr^2 - r^2(d\theta^2 + Sin^2(\theta)d\varphi^2) \tag{13}$$

with,

$$f(r) = 1 - \frac{2M}{r} + 2\beta\left(\frac{r^2\beta}{3} - \frac{1}{r}\int_r^\infty \sqrt{Q^2 + r^4\beta^2}\right) \tag{14}$$

In the limit $\beta \to \infty$, the elliptic integral can be expanded to give,

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \tag{15}$$

resulting the function $f(r)$ for the Reissner-Nordstrom black hole for Maxwell's electrodynamics.

The Hawking temperature of the black hole is given by,

$$T = \frac{1}{4\pi}\left[\frac{1}{r_+} + 2\beta\left(r_+\beta - \frac{\sqrt{(Q^2 + r_+^2\beta^2)}}{r_+}\right)\right] \tag{16}$$

Here, r_+ is the event horizon of the black hole which is a solution of $f(r) = 0$. The above black hole and its thermodynamic properties are discussed in [6]. Static charged black hole solution to the above action with a cosmological constant was presented in [36] and was extended to higher dimensions in [37].

3 Metric perturbations

In this section the equations for gravitational perturbations of charged black holes in Born-Infeld gravity are given. The notations of [38] are closely followed.

As described in section(2), Born-Infeld black hole is a stationary spherically symmetric time-independent solution of the field equations. Once perturbed, the metric will become non-stationary and time-dependent. Here, the perturbed metric will be

assumed to be axially symmetric. The most general metric which is time-dependent as well as axially symmetric can be written as,

$$ds^2 = e^{2\nu}dt^2 - e^{2\psi}(d\varphi - q_2 dx^2 - q_3 dx^3 - \omega dt)^2 - e^{2\mu_2}(dx^2)^2 - e^{2\mu_3}(dx^3)^2 \qquad (17)$$

Here, $\nu, \psi, \mu_2, \mu_3, \omega, q_2, q_3$ are functions of t, x^2, x^3 only. These functions do not depend on φ preserving axisymmetry.

The charged Born-Infeld black hole discussed in the previous section will be considered as a special, spherically symmetric time-independent solution of the above line element. Hence the unpertubed metric consists of the following functions;

$$e^{2\nu} = e^{-2\mu_2} = 1 - \frac{2M}{r} + 2\beta\left(\frac{r^2\beta}{3} - \frac{1}{r}\int_r^\infty \sqrt{Q^2 + r^4\beta^2}\right) = \frac{\triangle}{r^2} \qquad (18)$$

and

$$e^{\mu_3} = r, \quad e^{\psi} = rSin\theta, \quad \omega = q_2 = q_3 = 0, \quad \triangle = r^2 e^{2\nu} \qquad (19)$$

with the coordinates $x^2 = r$ and $x^3 = \theta$. Also the electrodynamic field strength F_{ab} and the Ricci tensor for the unperturbed metric has the following values;

$$F_{02} = -\frac{Q}{r^2} \qquad (20)$$

$$R_{00} = -R_{22} = -2\beta^2\left(1 - \frac{1}{\sqrt{1 - \frac{F_{02}^2}{\beta^2}}}\right) \qquad (21)$$

$$R_{11} = R_{33} = 2\beta^2\left(1 - \sqrt{1 - \frac{F_{02}^2}{\beta^2}}\right) \qquad (22)$$

$$R_{01} = R_{02} = R_{03} = R_{12} = R_{13} = R_{23} = 0 \qquad (23)$$

A general perturbation of the black hole will lead to non zero values of ω, q_2, q_3, $\delta\nu, \delta\mu_2, \delta\mu_3$ and $\delta\psi$. The perturbations leading to non-vanishing values of ω, q_2 and q_3 are called axial perturbations and the perturbations leading to increments to ν, μ_2, μ_3, ψ are called polar perturbations. This will be discussed in detail in the next section. The perturbed equations are obtained by linearising the Einstein and Born-Infeld's equation around the unperturbed metric given in eq.(13).

To obtain the linearised perturbation equations for the above black hole, it is convenient to use the tetrad formalism to the space-time metric in eq.(17). We will use the indices $a, b = 0, 1, 2, 3$ for the orthonormal basis and $\mu, \nu = 0, 1, 2, 3$ for the coordinate basis. To handle the black hole in this paper the coordinates are chosen as $x^0 = t, x^1 = \varphi, x^2 = r$ and $x^3 = \theta$. The tetrad basis e^a_μ is chosen such that $e^a_\mu e^b_\nu \eta_{ab} = g_{\mu\nu}$, where $\eta_{ab} = (1, -1, -1, -1)$. For the metric in eq.(17), the tetrads e^a_μ are given by,

$$e^{(0)}_\mu = (e^\nu, 0, 0, 0) \qquad (24)$$

$$e_\mu^{(1)} = (-\omega e^\psi, e^\psi, -q_2 e^\psi, -q_3 e^\psi) \tag{25}$$

$$e_\mu^{(2)} = (0, 0, e^{\mu_2}, 0) \tag{26}$$

$$e_\mu^{(3)} = (0, 0, 0, e^{\mu_3}) \tag{27}$$

Tensors in coordinate and orthonormal basis are related to each other by the tetrads. For example the field strength $F_{\mu\nu} = F_{ab} e^a_\mu e^b_\nu$.

3.1 Born-Infeld equations

As mentioned in the previous section, there are total of eight equations resulting from the Bianchi identities and the equations of motion for Born-Infeld electrodynamics. The four equations resulting from Bianchi identities $\bigtriangledown_{[\mu} F_{\nu\theta]} = 0$ are as follows,

$$\left(e^{\psi+\mu_2} F_{12}\right)_{,3} + \left(e^{\psi+\mu_2} F_{31}\right)_{,2} = 0 \tag{28}$$

$$\left(e^{\psi+\nu} F_{01}\right)_{,2} + \left(e^{\psi+\mu_2} F_{12}\right)_{,0} = 0 \tag{29}$$

$$\left(e^{\psi+\nu} F_{01}\right)_{,3} + \left(e^{\psi+\mu_3} F_{13}\right)_{,0} = 0 \tag{30}$$

$$\begin{aligned} &\left(e^{\nu+\mu_2} F_{02}\right)_{,3} - \left(e^{\nu+\mu_3} F_{03}\right)_{,2} + \left(e^{\mu_3+\mu_2} F_{23}\right)_{,0} \\ &= e^{\psi+\nu} F_{01} Q_{23} + e^{\psi+\mu_2} F_{12} Q_{03} - e^{\psi+\mu_3} F_{13} Q_{02} \end{aligned} \tag{31}$$

There are four equations resulting from $\bigtriangledown_\mu G^{\mu\nu} = 0$ in the orthonormal basis as follows,

$$\left(e^{\psi+\mu_3} G_{02}\right)_{,2} + \left(e^{\psi+\mu_2} G_{03}\right)_{,3} = 0 \tag{32}$$

$$-\left(e^{\psi+\mu_3} G_{23}\right)_{,2} + \left(e^{\psi+\mu_2} G_{03}\right)_{,0} = 0 \tag{33}$$

$$\left(e^{\psi+\nu} G_{23}\right)_{,3} + \left(e^{\psi+\mu_3} G_{02}\right)_{,0} = 0 \tag{34}$$

$$\begin{aligned} &\left(e^{\mu_2+\mu_3} G_{01}\right)_{,0} + \left(e^{\nu+\mu_3} G_{12}\right)_{,2} + \left(e^{\nu+\mu_2} G_{13}\right)_{,3} \\ &= e^{\psi+\mu_2} G_{02} Q_{02} + e^{\psi+\mu_2} G_{03} Q_{03} - e^{\psi+\nu} G_{23} Q_{23} \end{aligned} \tag{35}$$

Here, the partial derivative of a function g is given with the notation,

$$(g)_{,a} = \frac{\partial g}{\partial x^a} \tag{36}$$

The function Q_{AB} are given by,

$$Q_{A0} = \frac{\partial q_A}{\partial x^0} - \frac{\partial \omega}{\partial x^A} \quad \text{and} \quad Q_{AB} = \frac{\partial q_A}{\partial x^B} - \frac{\partial q_B}{\partial x^A} \quad (A, B = 2, 3) \tag{37}$$

In the two groups of equations, eq.(28) and eq.(32) can be ignored since they just provide integrability conditions for the two following equations.

In the orthonormal basis, the only non-zero components of G_{ab} and F_{ab} for the unperturbed metric are,

$$G_{02} = -\frac{Q}{r^2} \tag{38}$$

and

$$F_{02} = -\frac{Q}{\sqrt{r^4 + Q^2/\beta^2}} \tag{39}$$

Equations (29), (30) and (35) can be written in the spherical coordinates adopted for the black hole as,

$$\left(re^{\nu} F_{01} Sin\theta\right)_{,r} + re^{-\nu} F_{12,t} Sin\theta = 0 \tag{40}$$

$$re^{\nu} \left(F_{01} Sin\theta\right)_{,\theta} + r^2 F_{13,t} Sin\theta = 0 \tag{41}$$

$$re^{-\nu} G_{01,t} + \left(re^{\nu} G_{12}\right)_{,r} + G_{13,\theta} = -Q(\omega_{,2} - q_{2,0}) Sin\theta \tag{42}$$

Also the equations (33), (34) and (31) written in spherical coordinates simplifies to,

$$re^{-\nu} G_{03,t} = \left(re^{\nu} G_{23}\right)_{,r} \tag{43}$$

$$\delta G_{02,t} - \frac{Q}{r^2}\left(\delta\psi + \delta\mu_3\right)_{,t} + \frac{e^{\nu}}{rSin\theta}\left(G_{23} Sin\theta\right)_{,\theta} = 0 \tag{44}$$

$$\left[\delta F_{02} - \frac{Q}{r^2}\left(\delta\nu + \delta\mu_2\right)\right]_{,\theta} + (re^{\nu} F_{30})_{,r} + re^{-\nu} F_{23,t} = 0 \tag{45}$$

3.2 The perturbation of the Ricci tensor

To facilitate the explanation of the equations, the Ricci tensor components for the general metric in eq.(17) in the orthonormal basis is given here. Note that these are given in Chandrasekhar's book [38].

$$\begin{aligned}
R_{00} = &-e^{-2\nu}\left[(\psi + \mu_2 + \mu_3)_{,0,0} + \psi_{,0}(\psi - \nu)_{,0} + \mu_{2,0}(\mu_2 - \nu)_{,0} + \mu_{3,0}(\mu_3 - \nu)_{,0}\right] \\
&+ e^{-2\mu_2}\left[\nu_{,2,2} + \nu_{,2}(\psi + \nu - \mu_2 + \mu_3)_{,2}\right] + e^{-2\mu_3}\left[\nu_{,3,3} + \nu_{,3}(\psi + \nu + \mu_2 - \mu_3)_{,3}\right] \\
&- \frac{1}{2} e^{2\psi - 2\nu}\left[e^{-2\mu_2} Q_{20}^2 + e^{-2\mu_3} Q_{30}^2\right]
\end{aligned} \tag{46}$$

$$\begin{aligned}
R_{11} = &-e^{-2\mu_2}\left[\psi_{,2,2} + \psi_{,2}(\psi + \nu + \mu_3 - \mu_2)_{,2}\right] - e^{-2\mu_3}\left[\psi_{,3,3} + \psi_{,3}(\psi + \nu + \mu_2 - \mu_3)_{,3}\right] \\
&+ e^{-2\nu}\left[\psi_{,0,0} + \psi_{,0}(\psi - \nu + \mu_2 + \mu_3)_{,0}\right] + \frac{1}{2} e^{2\psi - 2\mu_2 - 2\mu_3} Q_{23}^2 \\
&- \frac{1}{2} e^{2\psi - 2\nu}\left[e^{-2\mu_3} Q_{30}^2 + e^{-2\mu_2} Q_{20}^2\right]
\end{aligned} \tag{47}$$

$$R_{22} = -e^{-2\mu_2}\left[(\psi+\nu+\mu_3)_{,2,2} + \psi_{,2}(\psi-\mu_2)_{,2} - \mu_{3,2}(\mu_3-\mu_2)_{,2} + \nu_{,2}(\nu-\mu_2)_{,2}\right]$$
$$-e^{-2\mu_3}\left[\mu_{2,3,3} + \mu_{2,3}(\psi+\nu+\mu_2-\mu_3)_{,3}\right] + e^{-2\nu}\left[\mu_{2,0,0} + \mu_{2,0}(\psi-\nu+\mu_2+\mu_3)_{,0}\right]$$
$$-\frac{1}{2}e^{2\psi-2\mu_2}\left[e^{-2\mu_3}Q_{23}^2 - e^{-2\nu}Q_{20}^2\right] \tag{48}$$
$$R_{01} = -\frac{1}{2}e^{-2\psi-\mu_2-\mu_3}\left[(e^{3\psi-\nu-\mu_2+\mu_3}Q_{20})_{,2} + (e^{3\psi-\nu-\mu_3+\mu_2}Q_{30})_{,3}\right] \tag{49}$$
$$R_{12} = -\frac{1}{2}e^{-2\psi-\nu+\mu_3}\left[\left(e^{3\psi+\nu-\mu_2+\mu_3}Q_{32}\right)_{,3} + \left(e^{3\psi-\nu+\mu_3-\mu_2}Q_{02}\right)_{,0}\right] \tag{50}$$
$$R_{02} = -e^{-\mu_2-\nu}\left[(\psi+\mu_3)_{,2,0} + \psi_{,2}(\psi-\mu_2)_{,0} + \mu_{3,2}(\mu_3-\mu_2)_{,0} - (\psi+\mu_3)_{,0}\nu_{,2})\right]$$
$$+\frac{1}{2}e^{2\psi-\nu-2\mu_3-\mu_2}Q_{23}Q_{30} \tag{51}$$
$$R_{23} = -e^{-\mu_2-\mu_3}\left[(\psi+\nu)_{,2,3} - \mu_{2,3}(\psi+\nu)_{,2} - \mu_{3,2}(\psi+\nu)_{,3} + \psi_{,2}\psi_{,3} + \nu_{,2}\nu_{,3}\right]$$
$$+\frac{1}{2}e^{2\psi-2\nu-\mu_2-\mu_3}Q_{20}Q_{30} \tag{52}$$

The other components R_{33}, R_{13} and R_{03} are not given here. They can be obtained by interchanging the indices 2 and 3 in R_{22}, R_{12} and R_{02}. The Ricci tensor for the Born-Infeld electrodynamics is given by,

$$R_{ab} = -2\left[\eta^{cd}\frac{F_{ac}F_{bd}}{\sqrt{1+\frac{F^2}{2\beta^2}}} + \eta_{ab}\left\{\beta^2\left(1-\sqrt{1+\frac{F^2}{2\beta^2}}\right)\right\}\right] \tag{53}$$

Since the expressions for the perturbed Ricci tensor is given in terms of the metric functions, one has to compute the changes to the Ricci tensor via the energy momentum tensor to obtain the complete equations. Therefore the perturbed components of the Ricci tensor for the Born-Infeld case are computed as follows;

$$\delta R_{ab} = -2\left[\frac{4F_{02}\delta F_{02}}{\sqrt{1-\frac{F_{02}^2}{\beta^2}}}\left(\frac{\eta_{ab}}{4} + \frac{\eta^{nm}F_{an}F_{bm}}{4\beta^2(1+\frac{F^2}{2\beta^2})}\right) + \eta^{nm}\left(\frac{\delta F_{an}F_{bm} + F_{an}\delta F_{bm}}{\sqrt{1+\frac{F^2}{2\beta^2}}}\right)\right] \tag{54}$$

Considering the fact that only non-zero component of F_{ab} before the perturbation is F_{02}, the exact expressions for δR_{ab} can be computed as,

$$\delta R_{00} = -\delta R_{22} = -\frac{2Q}{r^2}\frac{\delta F_{02}}{(1+\frac{F^2}{2\beta^2})} \tag{55}$$
$$\delta R_{11} = \delta R_{33} = -\frac{2Q}{r^2}\delta F_{02} \tag{56}$$
$$\delta R_{01} = -\frac{2Q}{r^2}\delta F_{12}, \quad \delta R_{03} = \frac{2Q}{r^2}\delta F_{23} \tag{57}$$
$$\delta R_{12} = \frac{2Q}{r^2}\delta F_{01}, \quad \delta R_{23} = \frac{2Q}{r^2}\delta F_{03} \tag{58}$$
$$\delta R_{13} = \delta R_{02} = 0 \tag{59}$$

3.3 Two categories of metric perturbations

The metric perturbations will lead to non-zero values of $\delta F_{02}, F_{03}, F_{23}, \delta\nu, \delta\psi, \delta\mu_2, \delta\mu_3$ and $\omega, q_2, q_3, F_{01}, F_{12}, F_{13}$. If only the terms in first order in perturbations are kept in the equations of motion, the above two groups of increments can be treated separately leading to two kinds of perturbations. To clarify this further, one can consider the perturbed Ricci tensors and make the analogy as follows; The components $R_{00}, R_{11}, R_{22}, R_{33}$ will change only if δF_{02} is non-zero. The components R_{03} and R_{23} will undergo changes only if F_{23} and F_{03} are non-zero respectively. By studying the expressions for the Ricci tensors in terms of the metric components, it is clear that this also means $\delta\nu, \delta\psi, \delta\mu_2, \delta\mu_3$ values has to be non-zero. On the other hand, the components R_{01}, R_{12} and R_{13} will change only if F_{12}, F_{01}, F_{13} are non-zero respectively also meaning ω, q_2, q_3 has to be non-zero. As explained in [38], the first group of rotations impart a rotation to the black hole while the second group does not. Hence the first is called *polar* and the second is called *axial* perturbations.

3.4 Axial perturbation

In this paper only the axial perturbations are considered. It is characterized by non-zero values of $\omega, q_2, q_3, F_{01}, F_{12}, F_{13}$. The two equations governing axial perturbations comes from R_{12} and R_{13}. Note that before perturbations, $R_{12} = R_{13} = 0$. After perturbations $\delta R_{13} = 0$ and $\delta R_{12} \neq 0$. By substituting the changes in the corresponding Ricci tensors and keeping the functions ν, ψ, μ_2, μ_3 as same as before the perturbations, we obtain the following equations,

$$\left(r^2 e^{2\nu} Q_{23} Sin^3\theta\right)_{,\theta} + r^4 Q_{02,t} Sin^3\theta = 2(r^3 e^{\nu} Sin^2\theta)\delta R_{12} = 4Qre^{\nu} F_{01} Sin^2\theta \tag{60}$$

and,

$$\left(r^2 e^{2\nu} Q_{23} Sin^3\theta\right)_{,r} - r^2 e^{-2\nu} Q_{03,t} Sin^3\theta = -2(r^2 Sin^2\theta)\delta R_{13} = 0 \tag{61}$$

Two new functions are defined as,

$$F_{01} Sin\theta = B, \quad G_{01} Sin\theta = \hat{B} \tag{62}$$

Considering the relation in eq.(4), B and $\hat{B}$ are related by $\hat{B} = B/p$ where,

$$p = \sqrt{1 + \frac{F^2}{2\beta^2}} \tag{63}$$

Eliminating G_{12} and G_{13} from eq.(42) with the help of eq.(40) and eq.(41) leads to,

$$\left[\frac{e^{2\nu}}{p}\left(re^{\nu}\hat{B}p\right)_{,r}\right]_{,r} + \frac{e^{\nu}}{r}\left(\frac{\hat{B}_{,\theta}}{Sin\theta}\right)_{,\theta} Sin\theta - re^{-\nu}\hat{B}_{,t,t} = Q(\omega_{,2,0} - q_{2,0,0})Sin^2\theta \tag{64}$$

By defining new functions $\hat{Q}(r,\theta,t)$ as,

$$\hat{Q}(r,\theta,t) = r^2 e^{2\nu} Q_{23} Sin^2\theta = \triangle(q_{2,3} - q_{3,2}) Sin^3\theta \tag{65}$$

eq.(60) and eq.(61) can be re written as,

$$\frac{1}{r^4 Sin^3\theta}\frac{\partial \hat{Q}}{\partial \theta} = -(\omega_{,2} - q_{2,0})_{,t} + \frac{4Q\hat{B}pe^{\nu}}{r^3 Sin^2\theta} \tag{66}$$

and

$$\frac{\triangle}{r^4 Sin^3\theta}\frac{\partial \hat{Q}}{\partial r} = (\omega_{,3} - q_{3,0})_{,t} \tag{67}$$

Taking the time dependence of the perturbed values of $\omega, q_2, q_3, \hat{Q}$ to be $e^{-i\omega t}$ where ω is the frequency of the modes, eq.(66) and eq.(67) are combined to be,

$$r^4 \frac{\partial}{\partial r}\left(\frac{\triangle}{r^4}\frac{\partial \hat{Q}}{\partial r}\right) + Sin^3\theta \frac{\partial}{\partial \theta}\left(\frac{1}{Sin^3\theta}\frac{\partial \hat{Q}}{\partial \theta}\right) + \frac{\omega^2 r^4 \hat{Q}}{\triangle} = 4Qe^{\nu} r \left(\frac{\hat{B}p}{Sin^2\theta}\right)_{,\theta} Sin^3\theta \tag{68}$$

Combining eq.(64) and eq.(66) leads to,

$$\left[\frac{e^{2\nu}}{p}(re^{\nu}\hat{B}p)_{,r}\right]_{,r} + \frac{e^{\nu}}{r}\left(\frac{\hat{B}_{,\theta}}{Sin\theta}\right)_{,\theta} Sin\theta + \left(\omega^2 r e^{-\nu} - \frac{4Qe^{\nu}p}{r^3}\right)\hat{B} = -Q\frac{\hat{Q}_{,\theta}}{r^4 Sin\theta} \tag{69}$$

An *ansatz* similar to the one given for the Reissner-Nordstrom black hole perturbation in [38] for the functions $\hat{Q}$ and $\hat{B}$ are made as,

$$\hat{Q}(r,\theta) = \hat{Q}(r) C_{l+2}^{-\frac{3}{2}}(\theta) \tag{70}$$

$$\hat{B}(r,\theta) = \frac{\hat{B}(r)}{Sin\theta}\frac{dC_{l+2}^{-\frac{3}{2}}}{d\theta} = 3\hat{B}(r) C_{l+1}^{-\frac{1}{2}}(\theta) \tag{71}$$

Here, C_n^{ν} denotes the Gegenbauer function which is a solution to the following differential equation,

$$\left[\frac{d}{d\theta} Sin^{2\nu}\theta \frac{d}{d\theta} + n(n+2\nu) Sin^{2\nu}\theta\right] C_n^{\nu}(\theta) = 0 \tag{72}$$

Gegenbauer function also satisfy the following recurrence relation,

$$\frac{1}{Sin\theta}\frac{dC_{\nu}^{n}}{d\theta} = -2\nu C_{n-1}^{\nu+1} \tag{73}$$

By substituting the functions $\hat{Q}(r,\theta)$ and $\hat{B}(r,\theta)$ in eq.(68) and eq.(69), the following two equations are obtained,

$$\triangle \frac{d}{dr}\left(\frac{\triangle}{r^4}\frac{d\hat{Q}}{dr}\right) - \mu^2 \frac{\triangle}{r^4}\hat{Q} + \omega^2 \hat{Q} = -\frac{4Q}{r^3}\mu^2 \triangle e^{\nu}\hat{B}p \tag{74}$$

and

$$\left[\frac{e^{2\nu}}{p}\left(re^{\nu}\hat{B}p\right)_{,r}\right]_{,r}-(\mu^2+2)\frac{e^{\nu}}{r}\hat{B}+\left(\omega^2 re^{-\nu}-\frac{4Q}{r^3}e^{\nu}p\right)\hat{B}=-Q\frac{\hat{Q}}{r^4} \tag{75}$$

Here,

$$\mu^2=2n=(l-1)(l+2) \tag{76}$$

The above eq.(74) and (75) are the main equations for the axial perturbations. The variable r will be replaced in terms of the "tortoise" coordinate defined by,

$$e^{2\nu}\frac{d}{dr}=\frac{d}{dr_*} \tag{77}$$

$\hat{Q}(r)$ and $\hat{B}(r)$ are redefined in terms of functions H_1 and H_2 as,

$$\hat{Q}=rH_2, \quad re^{\nu}\hat{B}p=-\frac{H_1 e^{-\frac{\phi}{2}}}{2\mu} \tag{78}$$

Here,

$$e^{\phi}=\frac{1}{p}\Rightarrow\phi=ln\left[\frac{\sqrt{Q^2+r^4\beta^2}}{r^2\beta}\right] \tag{79}$$

With these definitions, eq.(74) becomes,

$$\Lambda^2 H_2=\frac{\triangle}{r^5}\left\{\left[\mu^2 r-(e^{2\nu}r)'r+3e^{2\nu}r\right]H_2+2Q\mu e^{-\frac{\phi}{2}}H_2\right\} \tag{80}$$

and eq.(75) becomes,

$$\Lambda^2 H_1=\frac{\triangle}{r^5}\left\{\left[(\mu^2+2)r+\frac{4Q^2e^{-\phi}}{r}+r^3\left(\frac{(\phi' e^{2\nu})'}{2}+\frac{(\phi')^2e^{2\nu}}{4}\right)\right]H_2+2\mu Q e^{-\frac{\phi}{2}}H_1\right\} \tag{81}$$

Here,

$$\Lambda^2=\frac{d^2}{dr_*^2}+\omega^2 \tag{82}$$

As $\beta\to\infty$, both equations approaches the Reissner-Nordstrom black hole expressions given in [38].

3.5 Effective potentials

The above two equations can be considered as two one-dimensional wave equations coupled by the interaction matrix,

$$\left(\frac{d^2}{dr_*^2}+\omega^2\right)\begin{pmatrix}H_1\\H_2\end{pmatrix}=\begin{pmatrix}U_{11}&U_{12}\\U_{21}&U_{22}\end{pmatrix}\begin{pmatrix}H_1\\H_2\end{pmatrix} \tag{83}$$

where,

$$U_{11} = \frac{\triangle}{r^5}\left[(\mu^2+2)r + \frac{4Q^2e^{-\phi}}{r} + r^3\left(\frac{(\phi' e^{2\nu})'}{2} + \frac{(\phi')^2 e^{2\nu}}{4}\right)\right] \tag{84}$$

$$U_{12} = U_{21} = 2Q\mu\frac{\triangle}{r^5}e^{-\frac{\phi}{2}} \tag{85}$$

$$U_{22} = \frac{\triangle}{r^5}\left[\mu^2 r - (e^{2\nu}r)'r + 3e^{2\nu}r\right] \tag{86}$$

The matrix U can be diagonalized by a similarity transformation leading to one-dimensional Schrodinger type wave equation given by,

$$\Lambda^2 Z_i = V_i Z_i \quad (i=1,2) \tag{87}$$

where,

$$V_1 = \frac{1}{2}\left(U_1 + U_2 + \sqrt{(U_1-U_2)^2 + 4U_{12}^2}\right)$$
$$V_2 = \frac{1}{2}\left(U_1 + U_2 - \sqrt{(U_1-U_2)^2 + 4U_{12}^2}\right) \tag{88}$$

The perturbation equations of the Schwarzchild black hole were derived by Regge and Wheeler [39] and Zerilli [40]. The equations for the Reissner-Nordstrom black hole were first derived by Moncrief and Zerilli [42][41]. The Kerr black hole equations were derived by Teukolsky [43]. The equations for the charged black hole with a dilaton were derived in [44] [45].

The exact expressions for $V_{1,2}$ are not given here. However, the effective potential V_1 for the Born-Infeld black hole is plotted to show how it changes with charge Q and the non-linear parameter β in the following figures.

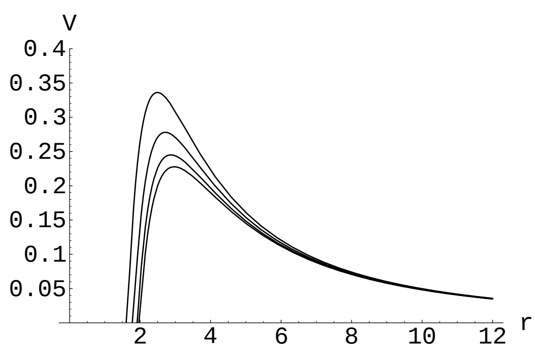

Figure 1. The behavior of the effective potential $V_1(r)$ with the charge for the Born-Infeld black hole. Here, $M = 1$, $\beta = 0.2$ and $l = 2$. The height of the potential decreases when the charge increases.

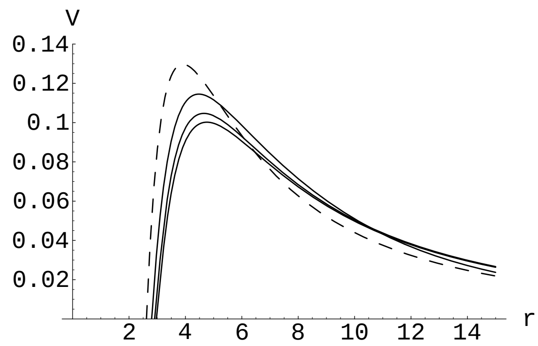

Figure 2. The behavior of the effective potential $V_1 r)$ with the non-linear parameter β. Here, $M = 1.5$, $Q = 1$ and $l = 2$. The maximum height of the potential increases as β increases. The dashed one is the potential for the Reissner-Nordstrom black hole with same mass and charge.

3.6 Remarks on stability

The potentials are real and positive out side the event horizon as it is evident from the above figures. Hence following the arguments by Chandrasekhar [38] the Born-Infeld black holes can be considered stable classically.

4 Quasi-normal modes of the Born-Infeld black hole

Quasi-normal modes (QNM) of a classical perturbation of black hole space-times are defined as the solutions to the related wave equations characterized by purely ingoing waves at the horizon. In addition, one has to impose boundary conditions on the solutions at the asymptotic regions as well. In asymptotically flat space-times, the second boundary condition is for the solution to be purely outgoing at spatial infinity. Once these boundary conditions are imposed, the resulting frequencies become complex and discrete.

Usually, the fundamental equation of black hole perturbations given in eq.(87) cannot be solved analytically. This is the case for almost all cases in 3+1 dimensions. In 2+1 dimensions there are two black hole solutions (BTZ black hole [46] and the charged dilaton black hole [47]) which can be solved to give exact values of QNM's. In five dimensions, exact values are obtained for vector perturbations by Nunez and Starinets [48].

There are several approaches to compute QNM's in literature. Here, a semi analytical technique developed by Iyer and Will [49] is followed. The method makes use of the WKB approximation, carried out to the sixth order. This approach has been

applied to the Schwarzchild [50] Reissner-Nordstrom [51], charged dilaton black hole [45] [31] [52]. The basics of this method is reviewed as follows;

Take the perturbation eq.(87) in the following form.

$$\left(\frac{d^2}{dr_*^2} + Q(r_*)\right) Z(r_*) = 0 \tag{89}$$

Here $Q(r_*) = \omega^2 - V_{1,2}(r_*)$. Then, one can define new variables $\Lambda(n), \Omega(n), \hat{\Lambda}(n), \hat{\Omega}(n), \alpha$ as follows.

$$\Lambda(n) = \frac{1}{(2Q_0^{(2)})^{1/2}} \left[\frac{1}{8} \left[\frac{Q_0^{(4)}}{Q_0^{(2)}} \right] (\frac{1}{4} + \alpha^2) - \frac{1}{288} \left[\frac{Q_0^{(3)}}{Q_0^{(2)}} \right]^2 (7 + 60\alpha^2) \right] \tag{90}$$

$$\Omega(n) = \frac{n + \frac{1}{2}}{2Q_0^{(2)}} \left[\frac{5}{6912} \left[\frac{Q_0^{(3)}}{Q_0^{(2)}} \right]^4 (77 + 188\alpha^2) - \frac{1}{384} \left[\frac{(Q_0^{(3)})^2 Q_0^{(4)}}{(Q_0^{(2)})^3} \right] (51 + 100\alpha^2) \right.$$
$$\left. + \frac{1}{2304} \left[\frac{Q_0^{(4)}}{Q_0^{(2)}} \right]^2 (67 + 68\alpha^2) + \frac{1}{288} \left[\frac{(Q_0^{(2)})^3 Q_0^{(5)}}{(Q_0^{(2)})^2} \right] (19 + 28\alpha^2) - \frac{1}{288} \left[\frac{Q_0^{(6)}}{Q_0^{(2)}} \right] (5 + 4\alpha^2) \right] \tag{91}$$

$$\alpha = n + \frac{1}{2}; \qquad \hat{\Lambda}(n) = -i\Lambda(n); \qquad \hat{\Omega}(n) = \Omega/(n + \frac{1}{2}) \tag{92}$$

Note that the superscript (n) denotes the appropriate number of derivatives of $Q(r_*)$ with respect to r_* evaluated at the maximum of $Q(r_*)$. In the case of black hole perturbations where $V(r_*)$ is independent of frequency ω, the quasi normal modes frequencies are given by,

$$\omega^2(n) = [V_0 + (-2V_0^{(2)})^{1/2} \hat{\Lambda}(n)] - i(n + \frac{1}{2})(-2V_0^{(2)})^{1/2}[1 + \hat{\Omega}(n)] \tag{93}$$

ω is represented as $\omega = \omega_R - i\omega_I$ and the lowest quasi normal modes $\omega(0)$ of the Born-Infeld black holes are computed.

First the quasi normal modes are computed to see the behavior with the non-linear parameter β as follows;

β	ω_R	ω_I
0.01	0.252195	0.035046
0.03	0.258583	0.037396
0.04	0.258527	0.035400
0.07	0.257616	0.034988
0.1	0.256993	0.035387
0.3	0.256105	0.035558
0.5	0.256011	0.035542

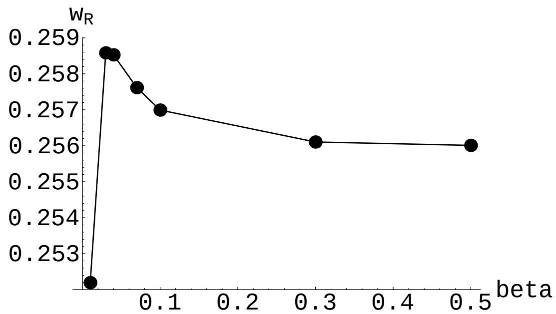

Figure 3. The behavior of Re ω with the non-linear parameter β for $M = 2$, $Q = 1$ and $l = 2$.

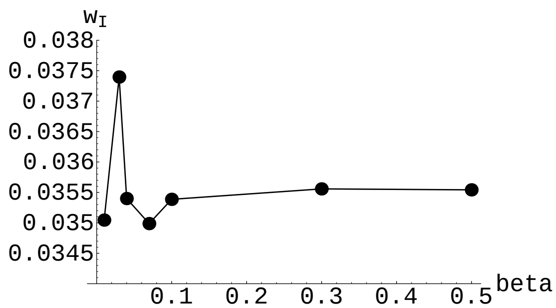

Figure 4. The behavior of Im ω with the non-linear parameter β for $M = 2$, $Q = 1$ and $l = 2$

The behavior of the quasi normal modes with varying charge Q was also studied as given in the following table.

Q	ω_R	ω_I
0.2	0.481698	0.068375
0.4	0.498791	0.069957
0.6	0.530753	0.072592
0.7	0.554740	0.074309
0.8	0.586874	0.076306
0.9	0.631678	0.078624
1	0.699423	0.081372

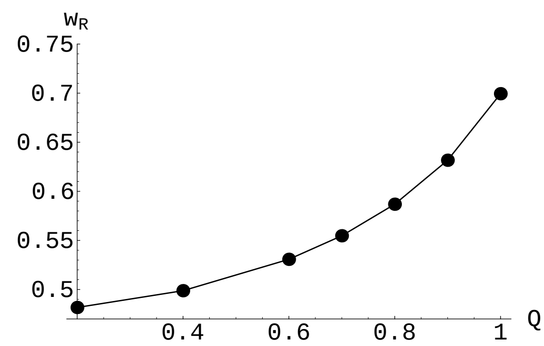

Figure 5. The behavior of Re ω with the charge Q for $M = 1$, $\beta = 0.4$ and $l = 2$

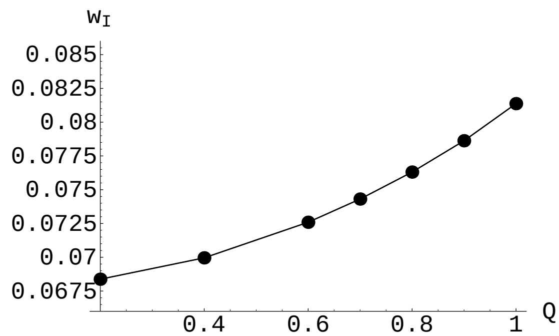

Figure 6. The behavior of Im ω with the charge Q for $M = 1$, $\beta = 0.4$ and $l = 2$

The behavior of the quasi normal modes of the Born-Infeld black hole is compared with the Reissner-Nordstrom black hole. The effective potential for the axial perturbation for the Reissner-Nordstrom black hole is computed in Chandrasekhar's book [38] which is given below.

$$V_i = \frac{(r^2 - 2Mr + Q^2)}{r^5}\left[(\mu^2+2)r - q_j\left(1 + \frac{q_i}{\mu^2 r}\right)\right], \quad (i,j = 1,2, i \neq j) \tag{94}$$

Here,

$$q_1 = 3M + \sqrt{9M^2 + 4Q^2\mu^2}; \quad q_2 = 3M - \sqrt{9M^2 + 4Q^2\mu^2} \tag{95}$$

The QNM's for V_1 is computed for varying charge and given below.

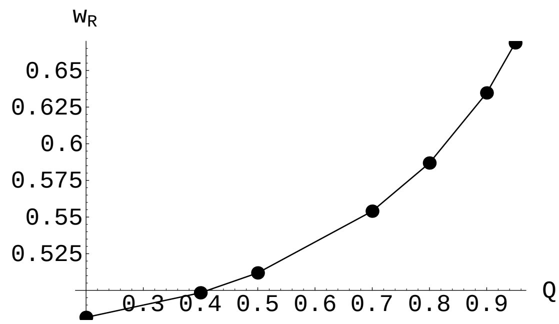

Figure 7. The behavior of Re ω with the charge Q for $M = 1$ and $l = 2$ for the Reissner-Nordsrom black hole

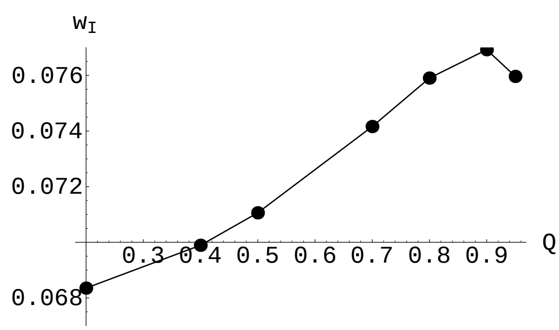

Figure 8. The behavior of Im ω with the charge Q for $M = 1$ and $l = 2$ for the Reissner-Nordsrom black hole

It is interesting to note that when the charge increases, the imaginary part of the QNM's continue to increase for the Born-Infeld black hole while for the Reissner-Nordstrom black hole it reaches a maximum and decreases.

The behavior of the quasi normal modes with spherical index l is given in the following table.

l	ω_R	ω_I
2	0.255970	0.035534
3	0.359628	0.039847
4	0.462125	0.042062
5	0.564053	0.043404
6	0.665646	0.044302
7	0.767021	0.044943

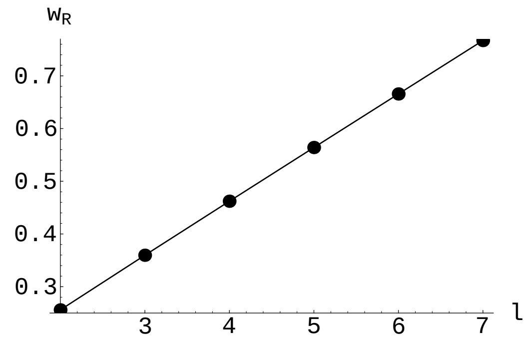

Figure 9. The behavior of Re ω with the the spherical index l for $M = 2$, $Q = 1$ and $\beta = 1$

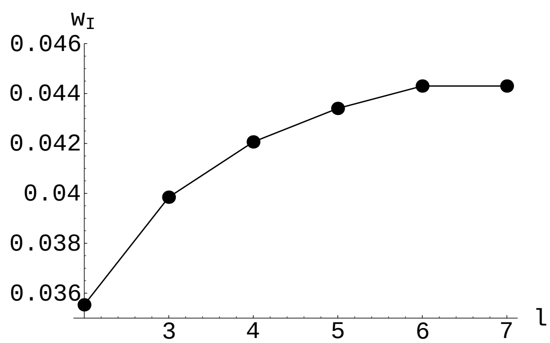

Figure 10. The behavior of Im ω with the spherical index l for $M = 2$, $Q = 1$ and $\beta = 1$

5 Conclusion

The gravitational perturbations of charged black holes in Einstein-Born-Infeld gravity are studied. The one dimensional Schrodinger type wave equations are derived for the axial perturbations. The behavior of the effective potential with the non-linear parameter β and charge Q are observed. From the behavior of the potentials it is concluded that the black holes are stable classically.

The lowest quasi-normal modes are computed using a WKB method. It was observed that when the β increases, the real part of QNM's increases to a maximum and decreases to a stable value. On the other hand, the imaginary part of QNM's shows an oscillating behavior before it stabilizers to a fixed value. Such behavior indicates that QNM's depend on β in a non trivial manner. As explained in [6] and [7], the number of horizons of the black hole depend on β for the same charge and mass. It is interesting to study how QNM's behavior changes depending on the horizon structure.

It was also observed that when charge increases, the QNM's increase. In comparison, for the Reissner-Nordstrom black hole the QNM's increases with the charge and decreases when the extremality reaches. I have not done a close study as to how the extreme nature of Born-Infeld black holes effects the behavior of the QNM's. This would be left for future work.

It is also noted that when the spherical index l is increased, the Re ω increases leading to greater oscillations. On the other hand Im ω approaches to a fixed value for larger l.

There are several avenues to proceed from here. One of the important extensions of this work is to compute the potentials and QNM's for the polar perturbations of the Born-Infeld black hole. It is well known that for the Schwarzchild and Reissner-Nordstrom black hole, the two potentials for axial and perturbations are related to each other in a simple manner leading to equality in reflexion and transmission coefficients. Hence they also have the same QNM's for both types of perturbations. Our expectation is that the modes would be *isospectral* for the Born-Infeld black hole as well. However, for the charged dilaton black hole, it was shown that the presence of the dilaton in fact breaks isospectrality [45].

It is interesting to investigate the supersymmetric nature of the Born-Infeld black hole discussed in this paper. It is well known that the extreme Reissner-Nordstrom black hole can be embedded in $N = 2$ supergravity theory [53] [54]. Onozawa et.al. [55] showed that the QNM's of the extreme RN black hole for spin 1, 3/2 and 2 are the same. If a suitable supergravity theory is constructed to embed the Born-Infeld black hole, such an investigation may support the supersymmetric nature of the black holes.

There are other approaches to calculate QNM's other than the method followed in this paper. For example power series expansion of the wave function is one of the methods in computing QNM's as used by Horowitz and Hubeny [21]. QNM's of higher modes are obtained with great precision with a semi-analytical method developed by Weaver [56]. This has been applied to Schwarzchild and Kerr black hole. It would be interesting to apply other methods to find QNM's and compare the results obtained in this paper.

From the Loop Quantum Gravity point of view, it is necessary to compute the QNM's for higher overtones. The method used in this paper is not accurate for large n. However, there are other approaches applied to compute highly damped QNM's [57] [58] [59] [60].

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